Integral geometry and separation of continuous and discrete spectra in regular representation of PSL (2, R)

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Dedicated to I.M. Gelfand
on his 75th birthday

Abstract. *The separation of continuous and discrete spectra in regular representation of PSL(2,* $\mathbb{R}$*) is constructed with methods of integral geometry.*

1. Let H be the Hilbert space of functions on the group $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm 1\}$ with norm

$$\|f\|^2 = \int |f(x)|^2 \, dx, \tag{1}$$

where dx is an invariant measure, and let T denote the regular representation of $PSL(2, \mathbb{R})$ in H:

$$(T(x)f)(y) = f(yx), \; x, y \in PSL(2, \mathbb{R}).$$

It is known [1] that T is decomposed into irreducible representations of continuous and discrete series. Thus H is direct sum of invariant subspaces, $H = H_c \oplus H_d$, where H_c is decomposed into representations of continuous series, and H_d into representations of discrete series. The problem is to construct this decomposition explicitly: The projectors $P_c : H \to H_c$, $P_d : H \to H_d$ are constructed

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in [2]. It turns out that methods of integral geometry allow one to construct simpler operators $J_c : H \to H_c$, $J_d : H \to H_d$, such that $(J_c)^2 = P_c$, $(J_d)^2 = P_d$. Let us formulate the main results of the paper. Later on we interpret H as the space of even functions on $SL(2, \mathbb{R}) = G$.

We define the following operators on the subspace $D \subset H$ of even C^∞ functions on G with compact support:

$$(J^{(\lambda)}f)(x) = \int_G f(y)\,|\,Tr\,(x^{-1}y)\,|^{\lambda}\,dy,$$

$$(\tilde{J}^{(\lambda)}f)\,(x) = \int_G f(y)\,|\,Tr\,(x^{-1}y\epsilon)\,|^{\lambda}\,dy, \qquad \epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\lambda \in \mathbb{C}$. Both integrals converge and are analytic functions of λ in the region $Re\lambda > 0$ for any $f \in D$, $x \in G$. For $Re\,\lambda < 0$ they are defined by analytic continuation w.r.t. λ.

We put $J_d = J^{(-2)}$, $J_c = \tilde{J}^{(-2)}$. Evidently all operators $J^{(\lambda)}$, $\tilde{J}^{(\lambda)}$, in particular J_d and J_c commute with representation operators $T(x)$.

THEOREM 1. *For any* $f \in D$ *we have:*

$$\|J_d f\|^2 + \|J_c f\|^2 = (2\pi)^4\,\|f\|^2,$$

where $\|\cdot\|$ *is the norm in* H.

COROLLARY. *The operators* J_d, J_c *can be extended to bounded operators on* H.

THEOREM 2. $J_d H = H_d$, $J_c H = H_c$ *(and so* $J_d H_c = 0$, $J_c H_d = 0$*).*

THEOREM 3. $(J_d)^2 = (2\pi)^4\,P_d$, $(J_c)^2 = (2\pi)^4 P_c$, *where* P_d *and* P_c *are projectors onto* H_d *and* H_c *respectively.*

Proofs are given in section 3.

2. Now we introduce the generalized Radon transform $\mathcal{R}$ related to the projective space $\mathbb{P}^3$ and establish its connection with operators J_c, J_d. Let F be the space of C^∞-functions on $\mathbb{R}^4 \setminus 0$ satisfying the homogeneity condition

(2) $$f(\lambda x) = \lambda^{-2} f(x) \ \text{ for any } \lambda \neq 0$$

Note [3] that F can be ingerpreted as the space of smooth sections of certain line bundle over $\mathbb{P}^3$.

We define the scalar product on F to be

$$(f_1, f_2) = \int_\Gamma f_1(x) \overline{f_2(x)}\, \omega(x),$$

where $\omega(x) = x_1\, dx_2 \wedge dx_3 \wedge dx_4 - x_2\, dx_1 \wedge dx_3 \wedge dx_4 + x_3\, dx_1 \wedge dx_2 \wedge dx_4 - x_4\, dx_1 \wedge dx_2 \wedge dx_3$ (*). The integral is over any surface $\Gamma \subset \mathbb{R}^4 \setminus 0$ meeting at one point almost every ray through 0; the homogeneity condition (2) implies that the integral is independent of Γ. Let $\bar{F}$ denote the Hilbert space which is the completion of F w.r.t. the norm $\|f\| = (f, f)^{1/2}$.

We introduce the symmetric bilinear form on $\mathbb{R}^4$ by

$$\langle x, y \rangle = x_1 y_4 - x_2 y_3 - x_3 y_2 + x_4 y_1$$

and define the generalized Radon transform of $f \in F$ to be

$$\mathscr{R} f(x) = (2\pi)^{-2} \int_\Gamma f(y) \langle x, y \rangle^{-2} \omega(y),$$

where the integral should be understood in regularized sense, see [3]. The integral does not depend on choice of Γ. It is easy to see that $\mathscr{R} f \in F$ (**).

It is known [3] that $\|\mathscr{R} f\| = \|f\|$ for $f \in F$, and that $\mathscr{R}^2 = id$, the identity operator. Therefore, $\mathscr{R} : F \to F$ is an isomorphism, and $\mathscr{R}$ extends to the involutive unitary operator on $\bar{F}$.

To find the connection of $\mathscr{R}$ with the operator J_c, J_d we use another realization of the space F. For this we interpret a point $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ as the matrix $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$; then $\langle x, x \rangle = 2 \det x$.

(*) In other words $\omega(x)$ is the internal product of the volume form $dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$ in $\mathbb{R}^4$ and the radial vector field $X = \Sigma\, x_i \partial/\partial x_i$.

(**) The ordinary Radon transform for $\mathbb{P}^3$ acts on the space of functions on $\mathbb{R}^4 \setminus 0$ which are homogeneous of degree (-3) as follows:

$$\mathscr{R} f(x) = \int_\Gamma f(y)\, \delta(\langle x, y \rangle)\, \omega(y),$$

where $\delta(\cdot)$ is the delta-function, see [3]; the choice of a bilinear form $\langle x, y \rangle$ is inessential. The connection of this transform with the classical Radon transform in three-dimensional affine space can also be found in [3].

Choose Γ to be the pair of surfaces $\langle x, x\rangle = 2$, and $\langle x, x\rangle = -2$, i.e. in our interpretation $\Gamma = G \cup \epsilon G$, $\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, is the variety of matrices with determinant ± 1. To each $f \in F$ we associate its restriction on Γ thus realizing F as the space of even functions on Γ. Note that $\omega(y)\,|_{\langle y,y\rangle = 2} = dy$ is an invariant measure on G, and that $\langle x, y\rangle = \pm\, Tr(yx^{-1})$ if $\det x = \pm 1$. It follows that in this new realization of F the norm and the operator $\mathcal{R}$ take the following form:

$$\|f\|^2 = \int_G |f(x)|^2\, dx + \int_G |f(\epsilon x)|^2\, dx, \tag{3}$$

$$\mathcal{R} f(x) = (2\pi)^{-2} \int_G f(y)\, (Tr(yx^{-1}))^{-2}\, dy + \tag{4}$$

$$+ (2\pi)^{-2} \int_G f(\epsilon y)\, (Tr(\epsilon y x^{-1}))^{-2}\, dy$$

We define the embedding $D \to F$, $H \to \overline{F}$ assigning to each $f \in H$ the function on Γ which is equal to $f(x)$ if $\det x = 1$, and equal to zero if $\det x = -1$. By (1) and (3) these embedding are isometric. By (4) for any $f \in D$ we have

$$\mathcal{R} f(x) = (2\pi)^{-2} \int_G f(y)\, (Tr(yx^{-1}))^{-2}\, dy, \quad \det x = \pm 1. \tag{5}$$

Hence by definition of J_c and J_d we have

PROPOSITION. $J_d f(x) = (2\pi)^2\, \mathcal{R} f(x)$, $J_c f(x) = (2\pi)^2\, \mathcal{R} f(\epsilon x)$ *for any* $f \in D$, $x \in SL(2, \mathbb{R})$. ■

3. Now we prove the theorems. Theorem 1 follows at once from (3), unitarity of $\mathcal{R}$ and the proposition just proven. In fact for $f \in D$ we have:

$$\|f\|^2 = \|\mathcal{R} f\|^2 = \int_G |\mathcal{R} f(x)|^2\, dx + \int_G |\mathcal{R} f(\epsilon x)|^2\, dx =$$

$$= (2\pi)^{-4} \left(\int_G |J_d f(x)|^2\, dx + \int_G |J_c f(x)|^2\, dx \right) =$$

$$= (2\pi)^{-4}(\|J_d f\|^2 + \|J_c f\|^2).$$

Theorems 2 and 3 follow from the next lemmas 1 - 3.

LEMMA 1. $(2\pi)^{-4}((J_d)^2 + (J_c)^2) = id.$

Proof. Let $f \in D$, $\varphi = \mathscr{R} f$. Since $\mathscr{R}^2 = id$, we have $f = \mathscr{R}\varphi$, i.e.

$$f(x) = (2\pi)^{-2}\int_G \varphi(y)\,(Tr(yx^{-1}))^{-2}\,dy +$$

$$+ (2\pi)^{-2}\int_G \varphi(\epsilon y)\,(Tr(\epsilon yx^{-1}))^{-2}dy =$$

$$= (2\pi)^{-4}\int_G J_d f(y)\,(Tr(yx^{-1}))^{-2}\,dy +$$

$$+ (2\pi)^{-4}\int_G J_c f(y)\,(Tr(\epsilon yx^{-1}))^{-2}\,dy =$$

$$= (2\pi)^{-4}\,(J_d^2 f(x) + J_c^2 f(x)).$$

LEMMA 2. $H_c \subset Ker\, J_d$.

Proof. Let L be the space of smooth functions on G with compact support such that $f(x\,k_\theta) = f(x)$ for any orthogonal matrix $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

To prove Lemma we shall use the following known property of H_c: $L \subset H_c$, and L is total in H_c i.e. finite linear combinations of the functions $T(x)f$, $x \in G$, $f \in L$, form a dense subspace of H_c. Therefore it is enough to verify that $L \subset Ker\, J_d$ i.e. that $J_d f = 0$ for any $f \in L$.

If $f \in L$ then

$$J^{(\lambda)}f(x) = \int_G f(y)\,|\,Tr(x^{-1}y)\,|^{\lambda}dy = \int_G f(y)\,|\,Tr(x^{-1}yk_\theta)\,|^{\lambda}dy$$

for any θ. Therefore,

$$J^{(\lambda)}f(x) = (2\pi)^{-1} \int_0^{2\pi} \int_G f(y) \,|\, Tr(zk_\theta) |^\lambda \, dy \, d\theta,$$

where $z = x^{-1}y = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$. Since $Tr(zk_\theta) = (z_1 + z_4) \cos\theta + (z_3 - z_2)$ $\sin\theta$, it follows that

$$\int_0^{2\pi} | Tr(zk_\theta) |^\lambda d\theta = \int_0^{2\pi} | (z_1 + z_4) \cos\theta + (z_3 - z_2) \sin\theta |^\lambda d\theta =$$

$$= [(z_1 + z_4)^2 + (z_3 - z_2)^2]^{\lambda/2} \int_0^{2\pi} | \cos\theta |^\lambda d\theta =$$

$$= 2\sqrt{\pi} \, \frac{\Gamma\left(\frac{\lambda+1}{2}\right)}{\Gamma\left(\frac{\lambda+2}{2}\right)} \, [(z_1 + z_4)^2 + (z_3 - z_2)^2]^{\lambda/2}.$$

Hence

$$(6) \qquad J^{(\lambda)}f(x) = \pi^{-1/2} \, \frac{\Gamma\left(\frac{\lambda+1}{2}\right)}{\Gamma\left(\frac{\lambda+2}{2}\right)} \int_G [(z_1 + z_4)^2 + (z_3 - z_2)^2]^{\lambda/2} f(y) dy$$

where $z = x^{-1}y$. Note that $(z_1 + z_4)^2 + (z_3 - z_2)^2 \geqslant 4$ on G and so the integral in (6) converges for any $x \in G$, $\lambda \in \mathbb{C}$. But

$$\left. \frac{\Gamma\left(\frac{\lambda+1}{2}\right)}{\Gamma\left(\frac{\lambda+2}{2}\right)} \right|_{\lambda=-2} = 0,$$

so $J_d f = J^{(-2)} f = 0$. ■

REMARK. Note that $J^{(\lambda)}f = 0$ for $f \in L$ and any $\lambda = -2k$, $k = 1, 2, \ldots$

Now we establish the connection of J_c with the operator of "horospheric transform"

$\mathscr{F}$ on G. By definition, $\mathscr{F}$ assigns to each function f on G its integrals over t two-sided classes (horospheres) $g_1^{-1} Z g_2$ where Z is the subgroup of matrices the form $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. If G is treated as the surface

$$x_1 x_4 - x_2 x_3 = 1$$

in $\mathbb{R}^4$ then the classes $g_1^{-1} Z g_2$ correspond to the line generators of tl surface; they have the equations

$$\alpha_1 x_1 + \alpha_2 x_3 = \beta_1$$

$$\alpha_1 x_2 + \alpha_2 x_4 = \beta_2,$$

where $(\alpha_1, \alpha_2) \neq 0$, $(\beta_1, \beta_2) \neq 0$. The operator $\mathscr{F}$ can be defined by

$$\mathscr{F} f(\alpha_1, \alpha_2; \beta_1, \beta_2) =$$

$$= \int_G f(x)\, \delta(\alpha_1 x_1 + \alpha_2 x_3 - \beta_1)\ \delta(\alpha_1 x_2 + \alpha_2 x_4 - \beta_2)\, dx,$$

where $\delta(\ )$ is the delfa function.

It is known [1] that $H_d \subset Ker\, \mathscr{F}$.

LEMMA 3. The operator J_c can be expressed as the composition $J_c = I \circ \mathscr{F}$ where $\mathscr{F}$ is the horospheric transform, and I acts on the functions $\varphi = \mathscr{F} f$

$$I\varphi(x) = \int_{-\infty}^{+\infty} \int_{-\pi/2}^{\pi/2} \varphi(e^{-t/2} \cos\theta, e^{-t/2} \sin\theta;$$

$$e^{t/2}(x_1 \cos\theta - x_3 \sin\theta),\ e^{t/2}(x_2 \cos\theta - x_4 \sin\theta) \cdot$$

$$\cdot\, sh^{-2} t \cdot cht\, d\theta\, dt, \qquad x \in G,$$

where the integral over t should be understood in the regularized sense.

The proof is straightforward.

COROLLARY. $H_d \subset Ker\, J_c$.

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